Robust Optimization Applied to a Currency Portfolio

R. Fonseca, S. Zymler, W. Wiesemann, B. Rustem

Workshop on Numerical Methods and Optimization in Finance

June, 2009

Outline

Motivation

- Impact of foreign currency in a portfolio with international assets
  - Asset gains may be overriden by an exchange rate depreciation
  - If correctly modelled, exchange rate appreciations may represent another source of profit
- FOREX market is one of the largest and most liquid financial markets
- In certain periods significant gains may be achieved by holding foreign currency (Levy 1978)
  - From Jan02 to Dec08, EUR, CHF and JPY had an average monthly appreciation rate against the USD of 0.46%, 0.45% and 0.43% respectively.
- The issue of hedging and finding the best financial instrument and the optimal hedge ratio — "Universal hedging" (Black 1989)

AIMS

- Develop a robust optimization approach for a currency-only portfolio
  - Formulation of the triangulation property to avoid non-convexity issues
- Develop a hedging strategy, using currency options, coupled with robust optimization
- Explore the relationship between robust optimization, particularly the size of the uncertainty sets, and hedge ratio defined as:

$$\frac{\text{foreign currency options}}{\text{holdings of foreign currency}}$$

Notation

- Portfolio with n different currencies
- E_i^0 and E_i are the today and future spot exchange rates respectively of the ith currency, expressed as the number of units of the domestic currency per unit of the foreign currency
- $e_i = E_i/E_i^0$ is the return on any currency i

Deterministic Model

$$\max_{w \in \mathbb{R}^n} \quad \mathbf{w}'\mathbf{e}$$

$$\begin{aligned} \text{s. t.} \quad \mathbf{w}'\mathbf{1} &= 1 \\ \mathbf{w} &\geq 0 \end{aligned}$$

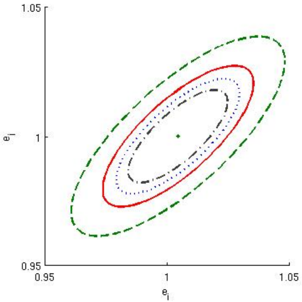
- Criticism: lack of robustness. Small deviations in the realised values of the returns from their estimates may render the solution infeasible.

UNCERTAINTY SETS

- Currency returns are random parameters expected to be in the future within some interval or uncertainty set, Θ_e:

$$\Theta_e = \{\mathbf{e} \geq 0 : (\mathbf{e} - \bar{\mathbf{e}})'\Sigma^{-1}(\mathbf{e} - \bar{\mathbf{e}}) \leq \delta^2\}$$

- Joint confidence region as deviations from the means are weighted by the covariance matrix.



Triangulation Property I

- Defining the exchange rates EUR/USD and GBP/USD implies determining the cross exchange rate EUR/GBP – *Triangulation Property*
- If this relationship does not hold at all times, a risk free profit can be made.
- A new constraint needs to be added to guarantee that no arbitrage is possible:

$$
\begin{aligned}
E_i \cdot \frac{1}{E_j} \cdot CE_{ij} &= 1, \quad \forall i, j = 1, \ldots, n \text{ and } (ij) = 1, \ldots, \frac{n(n-1)}{2} \\
\Leftrightarrow e_i \cdot \frac{1}{e_j} \cdot ce_{ij} &= 1
\end{aligned}
$$

- This however renders the model non convex.

Triangulation Property II

- Cross exchange rates do not impact the return of the portfolio, but only restrict the size of the uncertainty set of the exchange rates
- If the uncertainty of the cross exchange rates is modelled as an interval centered at the estimate, then:

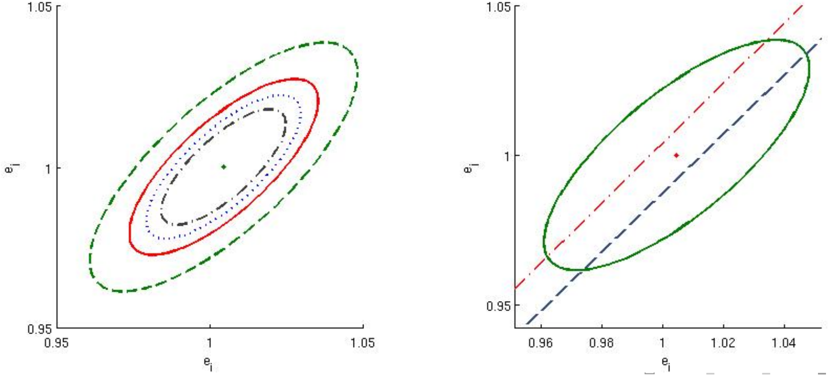
$$
\begin{aligned}
\underline{ce} \leq \quad & ce_{ij} \quad \leq \bar{ce} \\
\Leftrightarrow \underline{ce} \leq \quad & \frac{e_j}{e_i} \quad \leq \bar{ce} \\
\Leftrightarrow \underline{ce} \cdot e_i \leq \quad & e_j \quad \leq \bar{ce} \cdot e_i, \quad \forall e_i \neq 0
\end{aligned}
$$

- Convexity is preserved with $n(n-1)$ linear inequalities.

UNCERTAINTY SETS II

- Currency returns are random parameters expected to be in the future within some interval or uncertainty set, Θ_e:

$$\Theta_e = \{\mathbf{e} \geq 0 : (\mathbf{e} - \bar{\mathbf{e}})' \Sigma^{-1} (\mathbf{e} - \bar{\mathbf{e}}) \leq \delta^2 \wedge \mathbf{Ae} \geq 0\}$$



Robust Counterpart

- Maximization of the portfolio return for the worst case of the currency returns within the uncertainty set specified
- A is the coefficient matrix reflecting the triangular relationships among the exchange rates

Robust Formulation

$$\max_{w \in \mathbb{R}^n} \min_{e \in \Theta_e} \quad \mathbf{w}'\mathbf{e}$$

$$\begin{aligned} \text{s.t.} \quad \mathbf{Ae} &\geq 0, \quad \forall \mathbf{e} \in \Theta_e \\ \mathbf{w}'\mathbf{1} &= 1 \\ \mathbf{w}, \mathbf{e} &\geq 0 \end{aligned}$$

Robust Counterpart II

Solution Approach

- Problem is not yet in a form that may be passed on to a solver.

$$
\begin{aligned}
&\max_{w \in \mathbb{R}^n} \quad \phi \\
\text{s.t.} \quad & \mathbf{w}'\mathbf{e} \geq \phi, \quad \forall \mathbf{e} \in \Theta_e \\
& \mathbf{A}\mathbf{e} \geq 0, \quad \forall \mathbf{e} \in \Theta_e \\
& \mathbf{w}'\mathbf{1} = 1 \\
& \mathbf{w} \geq 0
\end{aligned}
$$

ROBUST COUNTERPART II

SOLUTION APPROACH

- Problem is not yet in a form that may be passed on to a solver.
- Start by solving the inner minimization problem with respect to the foreign exchange rate returns

$$\begin{array}{rrcll} & \max_{w \in \mathbb{R}^n} & \phi & & \\ \text{s.t.} & \mathbf{w}'\mathbf{e} & \geq & \phi, & \forall \mathbf{e} \in \Theta_e \\ & \mathbf{A}\mathbf{e} & \geq & 0, & \forall \mathbf{e} \in \Theta_e \\ & \mathbf{w}'\mathbf{1} & = & 1 & \\ & \mathbf{w} & \geq & 0 & \end{array}$$

$$\begin{array}{rrcl} & \min_{e \in \mathbb{R}^n} & \mathbf{w}'\mathbf{e} & \\ \text{s.t.} & \|\Sigma^{-1/2}(\mathbf{e} - \bar{\mathbf{e}})\| & \leq & \delta \\ & \mathbf{A}\mathbf{e} & \geq & 0 \\ & \mathbf{e} & \geq & 0, \end{array}$$

THE DUAL PROBLEM

- Duality theory states that in the case of second order cone programs, primal and dual problems have the same value of the objective function.

- Compute the dual problem:

$$\max_{v,s} \quad \bar{\mathbf{e}}'(\mathbf{w}-\mathbf{s}) - \delta v$$

$$\begin{aligned} \text{s. t.} \quad \|\Sigma^{1/2}(\mathbf{w}-\mathbf{s})\| &= v \\ \mathbf{s}, v &\geq 0 \end{aligned}$$

with $\mathbf{s} = \mathbf{A}'\mathbf{k} + \mathbf{y}$

THE DUAL PROBLEM

- Duality theory states that in the case of second order cone programs, primal and dual problems have the same value of the objective function.

- Compute the dual problem:

$$\max_{v,s} \quad \bar{\mathbf{e}}'(\mathbf{w}-\mathbf{s}) - \delta v$$

$$\begin{aligned} \text{s. t.} \quad \|\Sigma^{1/2}(\mathbf{w}-\mathbf{s})\| &= v \\ \mathbf{s}, v &\geq 0 \end{aligned}$$

with $\mathbf{s} = \mathbf{A}'\mathbf{k} + \mathbf{y}$

- Replace in the original problem:

$$\max_{w,s} \quad \phi$$

$$\begin{aligned} \text{s. t.} \quad \bar{\mathbf{e}}'(\mathbf{w}-\mathbf{s}) - \delta\|\Sigma^{1/2}(\mathbf{w}-\mathbf{s})\| &\geq \phi \\ \mathbf{w}'\mathbf{1} &= 1 \\ \mathbf{w}, \mathbf{s} &\geq 0 \end{aligned}$$

Hedging and Robust Optimization

- Non-inferiority property of robust optimization ensures that, unless the worst case scenario materializes, portfolio return will always be better than expected.
- For depreciations of the foreign exchange rates within the uncertainty set specified the investor is protected, without having to enter into any hedging agreement.
- An additional guarantee may be included to account for the possibility that the returns fall outside the uncertainty set.
- Currency options are added as an investment possibility to hedge against depreciations of the exchange rates outside the uncertainty set, complementing the robust optimization strategy.

Currency Options I

Characteristics

- Right, but not the obligation, to buy (*call*) or sell (*put*) an asset under specified terms: a maturity date, a strike price K and a premium p
- At maturity date (European options) the investor must decide on whether or not to exercise the option by comparing the spot exchange rate E with the strike price K:

$$\begin{aligned} V_{call} &= \max\{0, E - K\} \\ V_{put} &= \max\{0, K - E\} \end{aligned}$$

- Increased flexibility: downside risk protection while benefiting from upward movements in the price of the underlying asset

Currency Options II

Hedging

- By holding a long position on a put option, the investor is protected from decreases in the price of the underlying asset.
- The value of a portfolio which includes only one currency and a put option on that same currency is:

$$\begin{aligned} V_{port} &= E + \max\{0, K - E\} \\ &= \max\{E, K\} \end{aligned}$$

- The minimum exchange rate prevailing in the future is therefore defined by K.

The Robust Hedging Model I

Integrating Options

- We define the options returns e^d as:

$$e^d = f(e) = \max\left\{0, \frac{K - E^0 e}{p}\right\}$$

  or equivalently:

$$e^d = f(e) = \max\{0, a_p + b_p e\} \quad \text{with} \quad a_p = \frac{K}{p} \quad \text{and} \quad b_p = -\frac{E^0}{p}$$

- To account for the possibility of foreign exchange returns being outside the uncertainty set Θ_e, we complement our investment with currency options, guaranteeing a minimum return (defined by parameter ρ).

The Robust Hedging Model II

Formulation

- Maximization of the portfolio return for the worst case of the currency returns inside the uncertainty set Θ_e.
- Additional guarantee provided by the currency options when the foreign exchange returns fall outside the uncertainty set.
- Investment in options limited by the holdings in foreign currencies — hedging purpose only.

$$\max_{w,w^d} \min_{e,e^d} \quad \mathbf{w}'\mathbf{e}$$

$$\begin{aligned} \text{s.t.} \quad \mathbf{Ae} &\geq 0, \quad \forall \mathbf{e} \in \Theta_e \\ \mathbf{w}'\mathbf{e} + \mathbf{w^d}'\mathbf{e^d} &\geq \rho(\mathbf{w}'\mathbf{e}), \quad \forall \mathbf{e}, \mathbf{e^d} \geq 0, \mathbf{e^d} = f(\mathbf{e}) \\ \mathbf{w^d}/\mathbf{p} &\leq \mathbf{w}/\mathbf{E^0} \\ (\mathbf{w} + \mathbf{w^d})'\mathbf{1} &= 1 \\ \mathbf{w}, \mathbf{w^d}, \mathbf{e}, \mathbf{e^d} &\geq 0 \end{aligned}$$

The Robust Hedging Model III

Solution Approach

$$\max_{w,w^d} \quad \phi$$

$$\begin{aligned} \text{s.t.} \quad \mathbf{w}'\mathbf{e} &\geq \phi, \quad \forall \mathbf{e} \in \Theta_e \\ \mathbf{A}\mathbf{e} &\geq 0, \quad \forall \mathbf{e} \in \Theta_e \\ \mathbf{w}'\mathbf{e} + \mathbf{w^d}'\mathbf{e^d} &\geq \rho\phi, \quad \forall \mathbf{e}, \mathbf{e^d} \geq 0, \mathbf{e^d} = f(\mathbf{e}) \\ (\mathbf{w} + \mathbf{w^d})'\mathbf{1} &= 1 \\ \mathbf{w^d}/\mathbf{p} &\leq \mathbf{w}/\mathbf{E^0} \\ \mathbf{w}, \mathbf{w^d}, \mathbf{e}, \mathbf{e^d} &\geq 0 \end{aligned}$$

- Our formulation is however intractable: there are constraints on a infinite number of variables (e and e^d).

The Robust Hedging Model III

Solution Approach

$$
\begin{aligned}
\max_{w,w^d} \quad & \phi \\
\text{s. t.} \quad \mathbf{w}'\mathbf{e} &\geq \phi, \quad \forall \mathbf{e} \in \Theta_e \\
\mathbf{A}\mathbf{e} &\geq 0, \quad \forall \mathbf{e} \in \Theta_e \\
\mathbf{w}'\mathbf{e} + \mathbf{w^d}'\mathbf{e^d} &\geq \rho\phi, \quad \forall \mathbf{e}, \mathbf{e^d} \geq 0, \mathbf{e^d} = f(\mathbf{e}) \\
(\mathbf{w} + \mathbf{w^d})'\mathbf{1} &= 1 \\
\mathbf{w^d}/\mathbf{p} &\leq \mathbf{w}/\mathbf{E^0} \\
\mathbf{w}, \mathbf{w^d}, \mathbf{e}, \mathbf{e^d} &\geq 0
\end{aligned}
$$

$$
\begin{aligned}
\min_{e,e^d} \quad & \mathbf{w}'\mathbf{e} + \mathbf{w^d}'\mathbf{e^d} \\
\text{s. t.} \quad \mathbf{e^d} &\geq \mathbf{a_p} + \mathbf{b_p}\mathbf{e} \\
\mathbf{e}, \mathbf{e^d} &\geq 0
\end{aligned}
$$

- Our formulation is however intractable: there are constraints on a infinite number of variables (e and e^d).
- Following a similar procedure, we start by solving the inner minimization problem for the hedged return.

The Dual Problem

Also in linear programs, duality theory shows that both primal and dual problems have the same value of the objective function.

- Compute the dual problem:

$$\max_{t} \quad \mathbf{t}'\mathbf{a_p}$$

$$\begin{aligned} \text{s. t.} \quad \mathbf{w} + \mathbf{b_p'}\mathbf{t} &\geq 0 \\ \mathbf{t} &\leq \mathbf{w^d} \\ \mathbf{t} &\geq 0 \end{aligned}$$

The Dual Problem

Also in linear programs, duality theory shows that both primal and dual problems have the same value of the objective function.

- Compute the dual problem:

$$
\begin{aligned}
\max_{t} \quad & \mathbf{t}'\mathbf{a_p} \\
\text{s. t.} \quad & \mathbf{w} + \mathbf{b_p'}\mathbf{t} \geq 0 \\
& \mathbf{t} \leq \mathbf{w^d} \\
& \mathbf{t} \geq 0
\end{aligned}
$$

- Replace in the original problem:

$$
\begin{aligned}
\max_{w,w^d,t,s} \quad & \phi \\
\text{s. t.} \quad & \bar{\mathbf{e}}'(\mathbf{w}-\mathbf{s}) - \delta\|\Sigma^{1/2}(\mathbf{w}-\mathbf{s})\| \geq \phi \\
& \mathbf{t}'\mathbf{a_p} \geq \rho\phi \\
& \mathbf{w} + \mathbf{b_p'}\mathbf{t} \geq 0 \\
& \mathbf{t} \leq \mathbf{w^d} \\
& (\mathbf{w} + \mathbf{w^d})'\mathbf{1} = 1 \\
& \mathbf{w^d}/\mathbf{p} \leq \mathbf{w}/\mathbf{E^0} \\
& \mathbf{w}, \mathbf{w^d}, \mathbf{t}, \mathbf{s} \geq 0
\end{aligned}
$$

Numerical Results

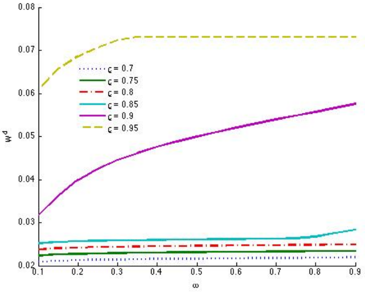
Data and Assumptions:

- Portfolio of six foreign currencies: EUR, GBP, JPY, CHF, CAD, AUD, measured against the USD.
- 50 alternative options were considered for each currency.
- Tested both models for different values of δ and ρ.
- We defined δ as $\sqrt{(1-\omega)/\omega}$, where $\omega \in [0,1]$.

Portfolio Composition:

- Portfolio composition changes in favour of the currencies that allow for the highest option returns.
- The highest strike price is chosen in order to minimize the difference between the worst case future spot exchange rate and the strike price, up to the desired amount of return guaranteed by ρ.

Relationship with ω and ρ



- Smaller uncertainty sets lead to higher investment in options ($\rho = 90\%$):

$$\omega = 10\% \Rightarrow W^d = 3\%$$

$$\omega = 90\% \Rightarrow W^d = 6\%$$

- Higher desired return in the extreme cases leads to higher investment in options ($\omega = 35\%$):

$$\rho = 70\% \Rightarrow W^d = 2\%$$

$$\rho = 95\% \Rightarrow W^d = 7\%$$

Simulated Backtesting

Backtesting procedure:

Step 1 Generate a 10-year time series of the monthly currency returns. Currencies are assumed to follow a geometric Brownian motion. Covariance matrix Σ and triangulation matrix $\mathbf{A}$ are assumed constant.

Step 2 Compute the estimated mean returns based on the previous twelve months.

Step 3 Compute the optimal portfolio weights and the realised portfolio return.

Step 4 Move forward one month and repeat Step 3, until the end of the time series.

Step 5 Compute the geometric mean return and the variance of the portfolio.

Step 6 Repeat Steps (1-5) 100 times with different random generator seeds.

Backtesting Results I

Robust Model vs Minimum Risk Model

- Robust Model:

ω(%)	M.Ret.	St.D.	Wins(%)
20	1.0033	0.0161	83
30	1.0034	0.0165	84
40	1.0035	0.0170	81
50	1.0036	0.0177	80
60	1.0036	0.0185	76
70	1.0037	0.0194	73

- Minimum Risk Model

M.Ret.	St.D.
1.0029	0.0156

- Smaller uncertainty sets lead to riskier portfolios.
- Robust model outperforms minimum risk model on over 70% of the cases on average.
- For $\omega = 30\%$, the % wins is at the maximum of 84%.

Backtesting Results II

Hedging Model vs Minimum Risk Model

- Hedging Model:

ω(%)	ρ(%)	M.Ret.	St.D.	Wins(%)
50	85	1.0035	0.0177	76
50	90	1.0035	0.0177	75
50	95	1.0038	0.0172	88
60	85	1.0035	0.0185	73
60	90	1.0036	0.0184	74
60	95	1.0041	0.0176	90
70	85	1.0036	0.0194	71
70	90	1.0036	0.0193	72
70	95	1.0044	0.0182	94

- Minimum Risk Model

1.0029	0.0156

- Good performance of the hedging model compared to the minimum risk model — over 70% Wins.
- Average monthly return of comparable performance relative to the robust model, with a slightly lower standard deviation.
- Greater monthly return when $\rho = 95\%$: higher return than the hedging model with a lower standard deviation.

CONCLUSION

We have seen:

- The need for including the triangulation property as a constraint in the robust model and how non-convexity issues may be solved by choosing appropriate uncertainty sets.
- An alternative hedging strategy using currency options together with a robust optimization approach.
- The impact of the size of the uncertainty sets in the total investment in options.
- Hedge ratios for the individual currencies are either 0 or 1 — investment in options is only up to the minimum amount that the investor wishes to guarantee.
- A cheaper hedging strategy by investing in out-of-the-money options with lower premiums.

Future Work

Future research will be directed towards the:

- Comparison of this approach with the alternative of hedging using forward exchange rates.
- Application of the hedging approach coupled with robust optimization to an international portfolio with assets and currencies.
- Extension to multi-stage problems:
  - Construction of scenario probabilities for both local asset and currency returns
  - Worst-case definition in terms of period? path?
  - Best hedging: rebalancing every period? are American options a viable alternative?