Bayesian Forecasting using an Extended Nelson-Siegel Model

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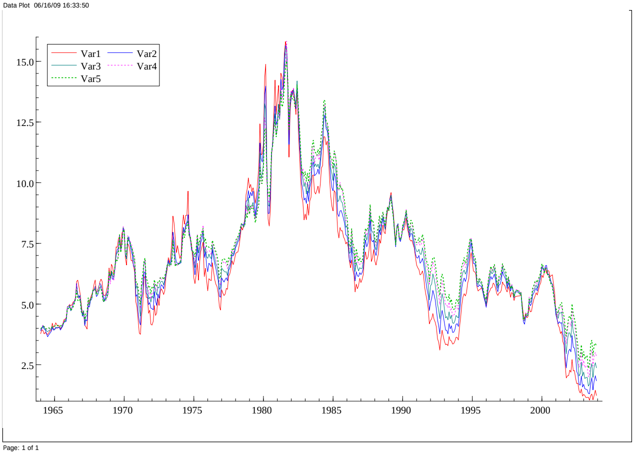
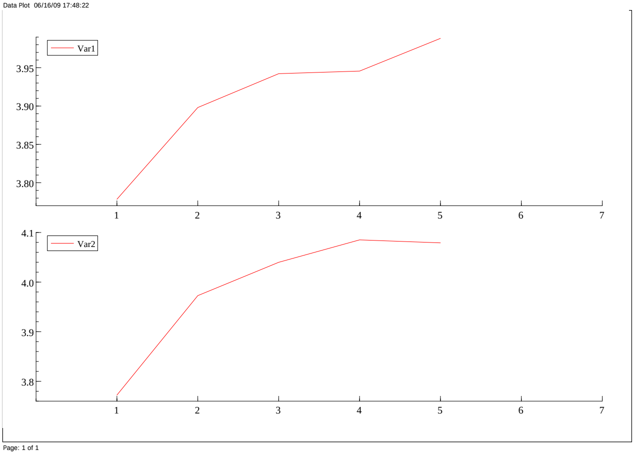
Plots of the Spot rates 1964:1-2003:12



Plots of the yield curve for 5 maturities



Motivations to Study Term Structure

- The yield curve carries information about prospective evolution of economic activity, inflation etc.
- Interest rate spreads (difference between 10 year and 3 month rates, or 10 year and 2 year rates) are predictive indicators of recession and inflation
- Since the yield curve assumes similar shape over time, we can explain the cross-sectional variation of interest rates through 3 factors: level, slope,and curvature.

The original NS model

The spot rate (yield)curve is expressed as:

$$y_t(n) = f_{1,t} + f_{2,t}\left[\frac{1-e^{-\lambda_t \cdot n}}{\lambda_t \cdot n}\right] + f_{3,t}\left[\frac{1-e^{-\lambda_t \cdot n}}{\lambda_t \cdot n} - e^{-\lambda_t \cdot n}\right]$$

- n: n period to maturity
- $\left[\ 1, \quad \frac{1-e^{-\lambda_t \cdot n}}{\lambda_t \cdot n}, \quad \frac{1-e^{-\lambda_t \cdot n}}{\lambda_t \cdot n} - e^{-\lambda_t \cdot n}\ \right]$: long term component, short term component and medium term component
- $[f_{1,t}, f_{2,t}, f_{3,t}]$: level, slope, curvature
- λ_t: governs the speed of decay in the short term component

The NS-DL model

$$\mathbf{y}_t = \mathbf{A}\mathbf{f}_t + \varepsilon_t \tag{1}$$

$$\mathbf{A} = \begin{pmatrix} 1 & \frac{1-e^{-\lambda_t \cdot 1}}{\lambda_t \cdot 1} & \frac{1-e^{-\lambda_t \cdot 1}}{\lambda_t \cdot 1} - e^{-\lambda_t \cdot 1} \\ 1 & \frac{1-e^{-\lambda_t \cdot 2}}{\lambda_t \cdot 2} & \frac{1-e^{-\lambda_t \cdot 2}}{\lambda_t \cdot 2} - e^{-\lambda_t \cdot 2} \\ \vdots & \vdots & \vdots \\ 1 & \frac{1-e^{-\lambda_t \cdot n}}{\lambda_t \cdot n} & \frac{1-e^{-\lambda_t \cdot n}}{\lambda_t \cdot n} - e^{-\lambda_t \cdot n} \end{pmatrix} : N \times 3$$

$$\mathbf{f}_t - \mu_{\mathbf{f}} = \Phi_{\mathbf{f}} \left(\mathbf{f}_{t-1} - \mu_{\mathbf{f}} \right) + \mathbf{u}_t \tag{2}$$

where $\varepsilon_t \overset{i.i.d}{\sim} f_{MN}\left(0_N, H_\varepsilon\right)$. $\mathbf{y}_t = \left\{ y_t\left(1\right), y_t\left(2\right), \ldots, y_t\left(N\right) \right\}'$.
$\mu_{\mathbf{f}} = \left\{ \mu_{1,f}, \mu_{2,f}, \mu_{3,f} \right\}'$, $\Phi_{\mathbf{f}} = \left\{ \phi_{1,f}, \phi_{2,f}, \phi_{3,f} \right\}'$

The NS-DL model

- $[f_{1,t}, f_{2,t}, f_{3,t}]$ are interpreted as latent dynamic factors
- NS-DL is able to capture the characteristics of term structure of interest rates over time

Can we learn about the volatilities of the dynamic factors?

The NS-SV model (Hautsch and Ou 2008)

Instead of assuming a constant variance-covariance matrix of $\mathbf{u}_t$
The NS-DL model is generalized to NS-SV to accommodate Stochastic Volatilities

$$\mathbf{u}_t = \begin{pmatrix} \sqrt{\exp h_{1,t}} & 0 & 0 \\ 0 & \sqrt{\exp h_{2,t}} & 0 \\ 0 & & \sqrt{\exp h_{3,t}} \end{pmatrix} \times \zeta_{j,t} \tag{3}$$

$$f_{j,t} - \mu_{j,f} = \phi_{j,f}\left(f_{j,t-1} - \mu_{j,f}\right) + \exp\left(\frac{h_{j,t}}{2}\right)\zeta_{j,t} : j = 1, 2, 3 \tag{4}$$

$$h_{j,t} - \mu_{j,h} = \phi_{j,h}\left(h_{j,t-1} - \mu_{j,h}\right) + \sigma_j \epsilon_{j,t} : j = 1, 2, 3 \tag{5}$$

where $\zeta_{j,t} \overset{i.i.d}{\sim} f_N(0,1)$ and $\epsilon_{j,t} \overset{i.i.d}{\sim} f_N(0,1)$

Estimation the model is challenging In Hautsch and Ou (2008), the model is estimated element by element using WinBugs

- The latent loading factors f and SV h are highly correlated, little movement in the draws (Chib and Greenberg 1996)
- chain converges slow and long chain is needed
- The estimate of $f_t \mid f_0, f_1, \ldots, f_{t-1}, f_{t+1}, \ldots, f_T$, and $h_t \mid h_0, h_1, \ldots, h_{t-1}, h_{t+1}, \ldots, h_T$
- $2 \times T$ steps are needed for each sweep, very time consuming

MCMC is a Solution to the Estimation problem

Target: random draws from $p(\theta \mid y)$ The MCMC can be constructed as follows:

- Partition θ into θ_1 and θ_2
- The posterior densities $p(\theta_1 \mid \theta_2, y)$ and $p(\theta_2 \mid \theta_1, y)$ have known analytical form
- generate random draws as the following:

$$\theta_1^n \sim p\left(\theta_1 \mid \theta_2^{n-1}, y\right)$$

$$\theta_2^n \sim p\left(\theta_2 \mid \theta_1^n, y\right)$$

Blocking the highly correlated latent variables

$$\begin{aligned}
\mathbf{y}_t &= \mathbf{A}\mathbf{f}_t + \varepsilon_t \\
\mathbf{f}_t - \mu_{\mathbf{f}} &= \Phi_{\mathbf{f}}\left(\mathbf{f}_{t-1} - \mu_{\mathbf{f}}\right) + \mathbf{u}_t \\
f^*_{j,t} &= h_{j,t} + z_{j,t} \\
h_{j,t+1} - \mu_{j,h} &= \phi_{j,h}\left(h_{j,t} - \mu_{j,h}\right) + \sigma_j \epsilon_{j,t} : j = 1, 2, 3
\end{aligned}$$

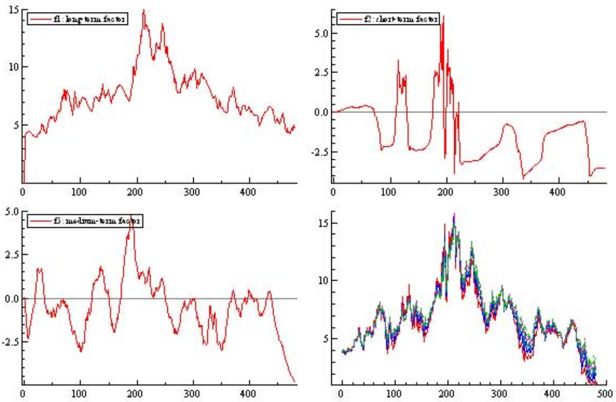
where

$$\begin{aligned}
f^*_{j,t} &= \ln\left\{\left[f_{j,t} - \mu_{j,f} - \phi_{j,f}\left(f_{j,t-1} - \mu_{j,f}\right)\right]^2 + c\right\} \\
z_{j,t} &= \ln\left(\zeta^2_{j,t}\right)
\end{aligned}$$

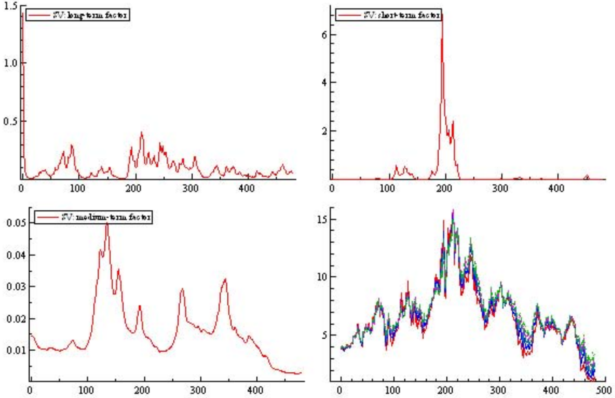
The improved method works

- Chain moves fast and algorithm is efficient according to the Relative Numerical Efficiency (RNE)
- Correlation among the draws are smaller
- latent variables are drawn in 6 blocks, only 6 steps are need in each iteration

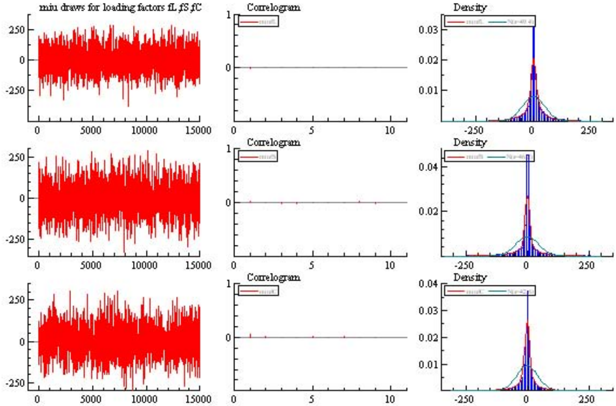
Estimated yield factors



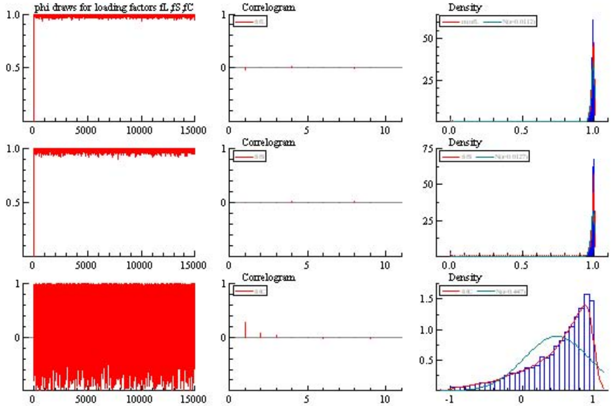
Filtered Stochastic Volatilities



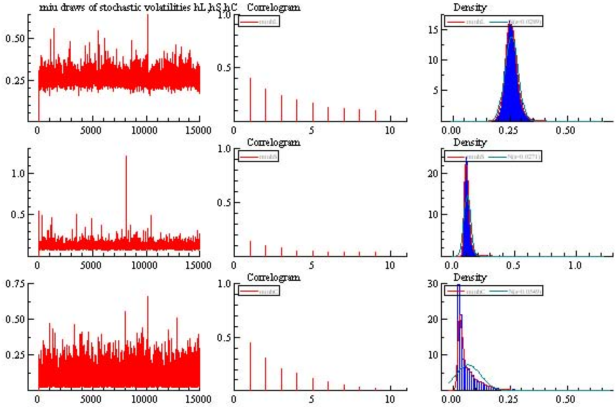
MCMC draws, Correlogram and Histogram for μ_f



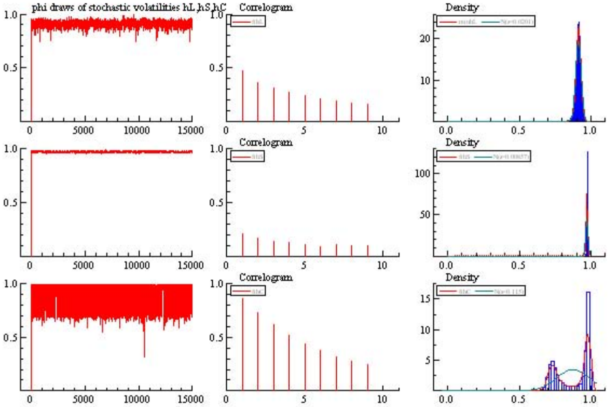
MCMC draws, Correlogram and Histogram for ϕ_f



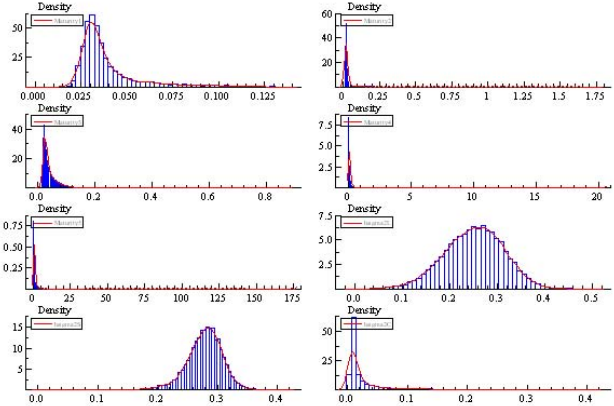
MCMC draws, Correlogram and Histogram for μ_h



MCMC draws, Correlogram and Histogram for ϕ_h



MCMC draws, Correlogram and Histogram for σ^2



MCMC draws, Correlogram and Histogram for λ

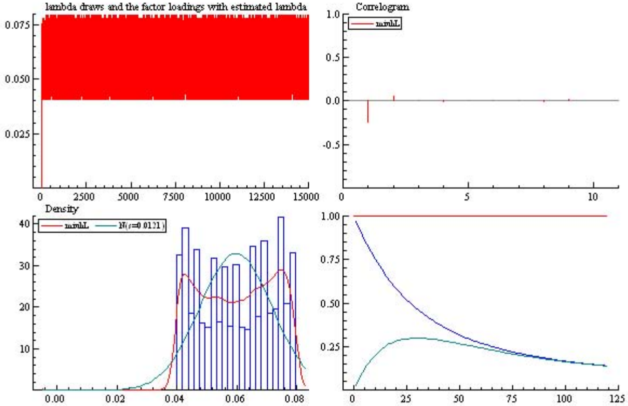


Table 1: Estimation results from MCMC

	$\phi_{f,L}$	$\phi_{f,S}$	$\phi_{f,C}$	$\mu_{h,L}$	$\mu_{h,S}$	$\phi_{h,L}$	$\phi_{h,S}$	$\phi_{h,C}$
mean	0.9954	0.9949	0.8647	0.2566	0.1123	0.9125	0.9738	0.8742
MCse	0.0066	0.0081	0.3020	0.0288	0.0271	0.0187	0.0032	0.1148
$HPIV_{lo}$	0.9818	0.9782	0.2071	0.2160	0.0848	0.8818	0.9680	0.7031
$HPIV_{up}$	0.9999	0.9999	0.9995	0.3030	0.1561	0.9412	0.9785	0.9905

,	$\sigma^2_{1,\varepsilon}$	$\sigma^2_{4,\varepsilon}$	$\sigma^2_{5,\varepsilon}$	$\sigma^2_{1,h}$	$\sigma^2_{2,h}$	$\sigma^2_{3,h}$	λ
mean	0.0386	0.0930	0.9758	0.2509	0.2789	0.0324	0.0598
MCse	0.0163	0.2572	3.1646	0.0619	0.0298	0.0490	0.0121
$HPIV_{lo}$	0.0243	0.0217	0.1417	0.1460	0.2268	0.0055	0.0407
$HPIV_{up}$	0.0756	0.2524	3.3381	0.3480	0.3228	0.1468	0.0780

CD, NSE, RNE

Table 2: Algorithm Evaluations 1

	$\mu_{f,L}$	$\mu_{f,S}$	$\mu_{f,C}$	$\phi_{f,L}$	$\phi_{f,S}$	$\mu_{h,L}$	$\mu_{h,S}$	$\mu_{h,C}$
cd	0.4583	−0.5092	−0.2334	0.4780	0.6622	0.3939	1.0604	0.0447
nse	0.4052	0.3776	0.3477	0.0001	0.0001	0.0002	0.0002	0.0004
$nse_{.04}$	0.3268	0.3830	0.3640	0.0001	0.0001	0.0007	0.0005	0.0008
$nse_{.08}$	0.2660	0.3845	0.3365	0.0001	0.0001	0.0007	0.0005	0.0007
$nse_{.15}$	0.2381	0.3518	0.2794	0.0001	0.0001	0.0007	0.0005	0.0006
$rne_{.04}$	1.5370	0.9720	0.9121	0.3538	0.3931	0.1072	0.1780	0.3111
$rne_{.08}$	2.3191	0.9643	1.0673	0.3941	0.4375	0.1300	0.1794	0.3805
$rne_{.15}$	2.8964	1.1520	1.5485	0.5450	0.3792	0.1238	0.2074	0.4856

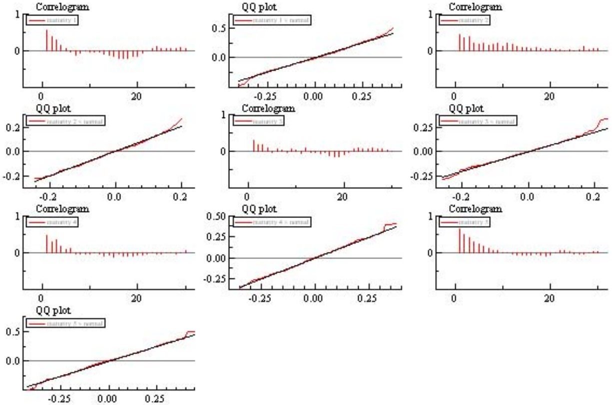
CD, NSE, RNE continue

Table 3: Algorithm Evaluations 2

	$\sigma^2_{1,\varepsilon}$	$\sigma^2_{3,\varepsilon}$	$\sigma^2_{4,\varepsilon}$	$\sigma^2_{5,\varepsilon}$	$\sigma^2_{1,h}$	$\sigma^2_{2,h}$	λ
cd	−0.4979	−0.6888	−0.8046	0.1797	−0.0070	−0.8839	1.0481
nse	0.0001	0.0002	0.0021	0.0259	0.0005	0.0002	0.0001
$nse_{.04}$	0.0002	0.0002	0.0019	0.0286	0.0010	0.0006	0.0001
$nse_{.08}$	0.0002	0.0002	0.0021	0.0272	0.0009	0.0005	0.0001
$nse_{.15}$	0.0002	0.0002	0.0023	0.0242	0.0008	0.0004	0.0001
$rne_{.04}$	0.5102	1.1849	1.1790	0.8210	0.2440	0.1636	1.9894
$rne_{.08}$	0.5212	1.5643	0.9900	0.9087	0.3000	0.2394	2.6511
$rne_{.15}$	0.6589	1.6099	0.8412	1.1508	0.4116	0.3651	3.3624

Model Fits check

Good model fits: Probability Integral Transforms (PIT) should be iid uniform.



Concluding Remarks

- Contributions:
  - Algorithm is efficient to estimate the NS-SV model
  - The iteration number in the MCMC can be reduced from 200,000,000 to 20,000.
  - NS-SV fits the data well
- Future research on the agenda:
  - Out-of-sample forecast with NS-SV
  - Term structure of interest rates and macrovariables
  - Model comparisons of NS class term structure model